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IV. *An Essay on the Connexion between the Parallaxes of the Sun and Moon; their Densities; and their disturbing Forces on the Ocean.* By Patrick Murdoch, D. D. F. R. S.

Read Feb. 18, 1768. **I**N a letter to Mr. Reid, formerly presented to the Hon. Society, and printed in Vol. LIV. Part II. of the Transactions, mention was made of a rule which I had used for computing the sun's parallax; but as that rule, though it gave a solution near the truth, was in part founded on authority (which, however respectable, ought to be cautiously admitted in such enquiries), I have considered the subject anew, upon such principles alone, as the established theory and the best observations furnished me. And the result is now humbly submitted to the Society.

I. The length of a second-pendulum at the equator, and on a level with the sea, being 36 inches $7\frac{1}{5}$ lines, Paris measure, according to the accurate observations of M. de la Condamine *, that length

* Journal du Voyage, &c. p. 163. In the copy of this paper, which was read in the Society, the length of a second-pendulum, at the equator, had been computed from its length at London: but here it is taken from the immediate observations of a very able astronomer. Some other small alterations have been made; and the examples placed in a better order.

(properly

(properly corrected) will, by the reasoning in the abovementioned letter, give the distance of a moon circulating round an unmoveable earth, equal to 59.95792 semidiameters of the equator. For the logarithm of this number, which is 1.7778438, write l .

Let L be the logarithm of some greater mean distance, inferred from observations of the moon's parallax; and if r be the natural number to the logarithm $3 \times \overline{L - l}$, and M be taken equal to $\frac{1}{r-1}$, the mass of the earth will be to that of the moon, as M to 1.

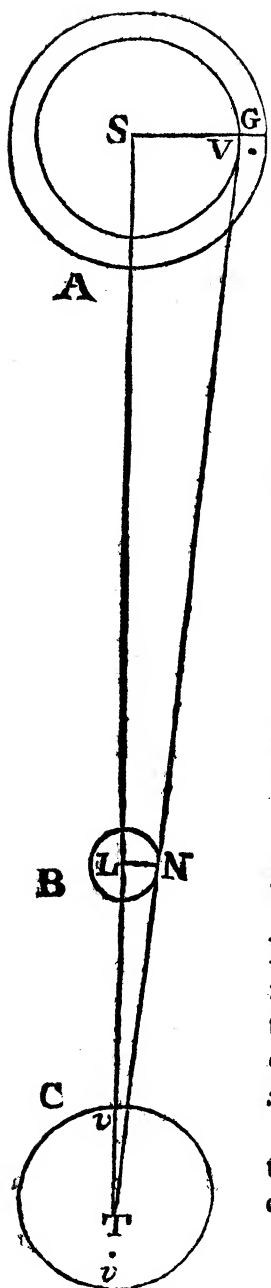
Conversely, If M be any how determined, its equal $\frac{1}{r-1}$, and r , with its logarithm $3 \times \overline{L - l}$ are known; $\frac{1}{3}$ of which is $\overline{L - l}$ to be added to l . For instance, if, with Sir Isaac Newton, we put $M = 39.788$, the distance will be 60.4557, the logarithm L being 1.7814372.

II. If, for each of these three, the moon's mass, her accelerative force on the earth, and her distance from the earth's centre, we write ($\phi =$) 1: the accelerative force of the earth on the moon will be represented by M , the mass just now computed. And if F is the sun's accelerative force on the earth, x his distance in semidiameters of the lunar orbit, Q the ratio of a sidereal year to a periodic month; we have (by Cor. 2. Prop. 4. Princip. I.) $\frac{F}{x} = \frac{M}{Q^2}$; a given ratio in given terms.

III. The terms F , x , therefore, must involve a common factor; by which being divided, the quote may be $\frac{M}{Q^2}$. And this might be obtained innumerable ways, were we to consider the ratio $\frac{F}{x}$ merely as an abstract quantity, altogether unrestricted: it were only putting $M^n \times Q^p = F$. And $\frac{Q^{2+p}}{M^{1-n}} = x$, or $M^{1-n} Q^p = F$, and $\frac{Q^{2+p}}{M^n} = x$: so as the sum of the *indices* of M should be unity, and the difference of those of Q should be 2.

But though the quantities F , x , are as yet unknown, they are not for that indeterminate and variable, as such a liberty of substitution would import: and all substitutions which imply the contrary, all *indices* which the theory disowns, or which are inconsistent with observation, are to be rejected. In a word, the *indices*, n , p , ought each of them to be *unique*, and determinate (*sine compare*), * as the quantities F and x are in nature. I take, therefore, $p = 1$, and $n = \frac{1}{2}$; that is, $F = M^{\frac{1}{2}} \times Q$, $x = \frac{Q^3}{M^{\frac{1}{2}}}$. See the examples in the table subjoined, upon different suppositions of the moon's distance.

* See Neut. Arith. Universal. in the Schol. to Prob. xxiv. The *maxima* and *minima* of variable quantities; the coordinates belonging to a double point, or to a point of reflexion, or contrary flexure, rays of curvature, limits of ratios, &c. All these are examples of the *unique*; that is, of quantities in a state that is distinguished from and exclusive of all others.



IV. The accelerative forces of two spherical bodies, A, B, upon a third C, are directly as their masses, and inversely as the squares of their central distances ST, LT, (or of x and d): which may be thus expressed, $\frac{F}{\phi} = \frac{A}{B} \times \frac{d^2}{x^2}$.

But the masses, A, B, being as their respective densities (which call s, m), and the cubes of (R, r) , the semidiameters of the spheres conjunctly; if we write for $\frac{A}{B}$

its equal, we have $\frac{F}{\phi} = \frac{s}{m} \times \frac{R^3}{r^3} \times \frac{d^2}{x^2}$.

Let the mass C be to B, as M to 1; and (f) its force on B, will be ϕM , or $\phi = \frac{f}{M}$, which gives $\frac{MF}{f} = \frac{s}{m} \times \frac{x}{d}$, and $\frac{F}{f} = \frac{\frac{s}{m} \times \frac{x}{d}}{\frac{f}{M}}$: supposing the apparent semidiameters of A and B to subtend the same angle at the centre of C, and thence R to be to r , as x to d .

V. But if SG, the semidiameter of A, be to the supposed semidiameter SV, as g to 1, then,

the density s remaining, the accelerative force of A (proportional to its magnitude) will be increased in the triplicate of that ratio, that is, we now have

$$\frac{F}{f} = \frac{q^3 s}{M m} \times x, \text{ putting } d = 1.$$

And the three bodies, A, B, C, representing the sun, moon, and earth; likewise Q being the ratio of the periods of the earth and moon, it is $\frac{F}{f} = \frac{x}{Q^2}$ (by the corollary quoted in Art. II.). Whence $\frac{s}{m} = \frac{M}{Q^3 \times Q^2}$.

VI. This accelerative force of A remaining, imagine the semidiameter S G to be reduced to its former magnitude S V; and the density of A will, at the same time, be increased to $s' = q^3 \times s$, and $\frac{s'}{m} = \frac{M}{Q^2}$.

In which case, namely, when the apparent semidiameters of A and B (the sun and moon) are equal, their powers to raise a tide at v , a vertex of C, will be as the densities s' , m *: that is, as M the ratio of the earth's mass to the moon's, and Q^2 the duplicate ratio of the year and month.

Or thus: The distances of the bodies, A, B, from the third C, being very great, their powers to raise a tide at v , or their disturbing forces on the ocean, will be directly as their accelerative forces at the centre of C, and inversely as their distances from it: that is, writing a , b , for the disturbing forces respectively; and for the sun's distance in semidiameters of C, the Letter z , it will be $a : b :: \frac{F}{z} : \frac{\phi}{d}$.

* See the last page of Dr. Sanderfon's Fluxions: or the late ingenious T. Simpson's Miscellaneous Tracts, p. 13.

For, by the general law, a is to F , as the difference of the squares of z and $z-1$ is to the square of $z-1$; that is, as $2z-1$ to z^2-2z+1 . And the same way b is to ϕ , as $2d-1$ to d^2-2d+1 ; whence, halving the antecedents, and retaining only z^2 , d^2 , in the consequents, we have the ratio of a to b , as above.

If the disturbing force of B is exerted at v^1 , the opposite vertex of C , b will now be to ϕ , as $2d+1$ to d^2+2d+1 ; and, in strictness, we ought to take a mean value of b : but this may be neglected as inconsiderable.

Lastly, let the sun's distance be again expressed in semidiameters of the lunar orbit; that is, if for z we write dx , and unity for ϕ , we have $a:b::\frac{F}{dx}:\frac{1}{d}$, or as M to Q^2 , as before.

VII. In Art. V. it was found that m denoting the density of the moon, and s that of the sun, q^3 being triple the ratio of the sun's mean semidiameter to the moon's *, then will $\frac{m}{s} = \frac{Q^2 q^3}{M}$. Whence it will easily follow, that the density of the earth is to that of the sun, as $Q^2 \times S^3$ to P^3 ; P being the moon's horizontal parallax, and S the sun's apparent semidiameter.

* In the *Principia*, the semidiameters are $16'. 6''$, and $15'. 38\frac{1}{4}''$; giving 0.0379755 for the logarithm of q^3 . Others take a few seconds from each; which does not much alter the value of q^2 .

VIII. I have applied these rules (as in the following Table) to the principal hypothesis of the moon's mean distance.

- 1°. Supposing it of 60.24 (from Dr. M. Stewart.)
 2°. ————— 60.4 (Cor. 7. Prop. 37. Princip. III.)
 3°. ————— 60.455 (Neut. M being 39.788.)
 4°. ————— 60.493 { (As by Mr. Short's calculations
 from the transit of ♀.)

The Moon's Mean Distance, being in Semidiameters of the Equator.

		60.24	60.4	60.455	60.493
		" "	" "	" "	" "
I.	Parall. ☽	57.4, 17	56.55 $\frac{1}{12}$	56.52	56.49, 88
II.	Mafs ☽	70.4225	44.823	39.788	36.9908
III.	☉ dist. to ☽	284.723	356.885	378.293	392.854
IV.	Parall. ☉	12 $\frac{1}{37}$	9, 58	9, $\frac{1}{2}$	8.69
V.	Denf. ☽	2, 77 : 1	4, 35 : 1	4, 9 : 1	5, 273 : 1
VI.	Denf. ☽	1, 449 : 1	0.9295 : 1	0.827 : 1	0.7707 : 1
VII.	Denf. ☽	4.0128 : 1	4.045 : 1	4.0559 : 1	4.06375 : 1
VIII.	Tides ☽	2.5379 : 1	3.9873 : 1	4.5544 : 1	4.8316 : 1
IX.	F. ☉ on ☽ F. ☽ on ☽	1.593 : 1 1.9968 : 1	1.9968 : 1 2.1194 : 1	2.1194 : 1 2.1981 : 1	2.1981 : 1 2.2198 : 1

R E M A R K S.

1. If it should be thought that the reasoning in Art. III. rests too much upon a metaphysical principle of Leibnitz, and requires, if not an apology, at least a more formal proof: the ground of such reasoning, and its extensive use, may be more particularly explained on some other occasion. Suffice it at present to add to the note on that article,

That as the given factors (Q and Q^3) in F and x , may be joined with M^n , or with M^{1-n} *indifferently*, the case is similar to that of an equal chance at play, for the stakes n and $1-n$, where the just expectation of each gamester is $\frac{1}{2}$, whatever be the value of n . The fifth and sixth Propositions of Element I. scarce needed any other demonstration than, that it is manifestly impossible to assign any *reason* of inequality, of the angles in one of these theorems, and of the sides in the other. In the following Proposition, where it is proved, that two lines being extended from the extremities of a right line, AB , to a point C , there is no other point on the same side of AB , to which lines from A and B equal to the former can be drawn: he who holds the contrary is supposed to fix upon that other point D ; but why D ? rather than d , d' , d'' , &c. he is silent: and, therefore, I conclude, there is no such point different from C . And the like may be said of some other simple theorems, that are commonly demonstrated by shewing the absurdity of asserting their contraries.

2. The title of this paper renders it almost needless to remind the reader, that the moon's parallax is not here proposed as the properest *medium* for determining that of the sun. Our *data* are still too uncertain for that purpose, scarce one of them having been determined to an unexceptionable precision; and the numbers in the table shew how much a small difference in the moon's distance must affect the several conclusions. It may be of use, however, to know in what manner those conclusions, as well as the quantities from which they derive *, stand related to one another. For if, hereafter, the necessary *data* should be more exactly known, the calculus may be repeated; and if the transit of Venus, which is to happen in 1769, should confirm Mr. Short's calculations from that of 1759, we may thence conclude the true mean distance of the moon, better than in any other way.

3. In the mean time, if any person should have the curiosity to examine the numbers of the table, he will please to take notice :

That, as no two measurements, nor any two lengths of a second-pendulum hitherto observed, make the earth of the same spheroid figure, I have retained for the ratio of its greatest and least diameters, that of 231 to 230 ; answering to the hypothesis of its uniform density : and have thence made a degree of the equator equal to 57200 French toises †.

* The connexions of F, π , Q, are manifest ; and the relation of M to Q^2 is easily deduced from Prop. 59. Princip. Book I.

† This was computed upon the supposition, that a degree of the meridian at lat. $49^{\circ}\frac{1}{3}$ is 57183 toises : but if that degree is, by
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The measure of a second-pendulum, at the equator, I had from M. de la Condamine, as was said before : it was corrected for the centrifugal force, and the resistance of the air ; and the moon's distance, as revolving round the earth at rest, was corrected for the sun's disturbing force : which is done either by diminishing that distance in the subtriplicate ratio of 178725 to 177725 ; or by diminishing the length of the pendulum in the simple ratio of these numbers. In this last correction I was favoured with the advice and assistance of the Rev. Mr. Price, and of the Astronomer Royal, FF. R. S.

a correction of M. Picart's operations, found to be only 57075 toises ; the diameter of the equator here used ought to be diminished by about 18 10000th parts.

E R R A T A.

- P. 7. l. 10. *for* confusion *read* concussion.
P. 28. l. 8 *for* $Q^3 \times Q^2$ *read* $q^3 \times Q^2$.
P. 29. *in the note, for* 00379755 *read* 90379755.
P. 30. l. 2. *for* hypotheis *read* hypotheses.
Ibid. throughout the table, for δ *read* δ .
P. 32. l. 16. *for* 1759 *read* 1761.
P. 175. l. 15. *for* being any number *read* z being any number.
P. 176. l. 4. *for* tohat *read* to that.
P. 179. l. 18. *for* $\frac{B}{\frac{1}{3}}$ *read* $\frac{B}{2\frac{1}{3}}$
P. 203. l. 24. *for* thing thing *read* thing.